

## Chapter 11. Infinite Sequences and Series.

### Section 11.1 - Sequences.

A sequence is an ordered list of numbers :  $a_1, a_2, \dots, a_n, a_{n+1}, \dots$

$a_n$  is called the  $n^{\text{th}}$  term of the sequence. The infinite sequence above is often denoted by  $\{a_n\}$ , or more specifically  $\{a_n\}_{n=1}^{\infty}$ .

Example ① the  $n^{\text{th}}$  term of the sequence  $\{1, 2, 3, 4, \dots\}$  is  $a_n = n$ . So we write  $\{1, 2, 3, 4, \dots\} = \{n\}_{n=1}^{\infty}$

② The  $n^{\text{th}}$  term of the sequence  $\{2, 5, 10, 17, 26, 37, 50, \dots\}$  is  $a_n = n^2 + 1$ . So we may denote the sequence by  $\{n^2 + 1\}_{n=1}^{\infty}$ .

③ The terms of the sequence  $\{1, -2, 4, -8, 16, -32, 64, \dots\}$  alternate in sign, and are integer (positive) powers of 2; That is  $a_n = (-1)^n \cdot 2^n$ , for  $n \geq 0$ . So we denote this sequence by  $\{(-1)^n \cdot 2^n\}_{n=0}^{\infty}$ .

④ What is the  $n^{\text{th}}$  term of  $\{-\frac{2}{2}, \frac{5}{3}, -\frac{10}{4}, \frac{17}{5}, -\frac{26}{6}, \dots\}$ ? The alternating sign suggests a factor of  $(-1)^n$ . The increasing denominator is modeled by  $(n+1)$ , and the numerators are (as in example 2) of the form  $(n^2 + 1)$ . Therefore, we have  $a_n = (-1)^n \cdot \frac{n^2 + 1}{n + 1}$ , for  $n \geq 1$ .

Remark: not all sequences are generated by explicit formulas for  $a_n$ . For example, the  $n^{\text{th}}$  term of the sequence  $\{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \dots\}$  is the  $n^{\text{th}}$  digit of  $\pi$ .

Remark: some sequences are defined recursively.

Example:  $a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$  for  $n \geq 3$  is the sequence

$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$  : The Fibonacci sequence.

**Definition:** A sequence  $\{a_n\}_{n=1}^{\infty}$  is convergent (or converges) if  $\lim_{n \rightarrow \infty} a_n$  exists (as a finite number). Otherwise the sequence is divergent (or diverges).

**Example ⑤** the sequence  $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$  converges since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

**⑥** the sequence  $\left\{\frac{n^2+1}{2n^2-1}\right\}_{n=1}^{\infty}$  converges to  $\frac{1}{2}$  since  $\lim_{n \rightarrow \infty} \frac{n^2+1}{2n^2-1} = \frac{1}{2}$

**⑦** the sequence  $\left\{\left(\frac{1.01}{100}\right)^n\right\}_{n=1}^{\infty} = \{1.01, 1.01^2, 1.01^3, \dots\}$  diverges since  $\lim_{n \rightarrow \infty} (1.01)^n = \infty$ .

**⑧** the sequence  $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{\infty} = \{1, 0, -1, 0, 1, 0, -1, 0, \dots\}$  diverges since  $\lim_{n \rightarrow \infty} \sin\left(\frac{n\pi}{2}\right)$  DNE

**⑨** the sequence  $\left\{\cos\left(\frac{2\pi}{n}\right)\right\}_{n=1}^{\infty}$  converges since  $\lim_{n \rightarrow \infty} \cos\left(\frac{2\pi}{n}\right) = \cos(0) = 1$ .

**Theorem:** If  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

This theorem is useful when dealing with alternating sequences.

**Example ⑩** Evaluate  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{2^n+1}$  if it exists. Here  $a_n = \frac{(-1)^n}{2^n+1}$ , so  $|a_n| = \frac{1}{2^n+1}$ .

Since  $\lim_{n \rightarrow \infty} \frac{1}{2^n+1} = 0$ , then  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(-1)^n}{2^n+1} = 0$  as well.

**Theorem:** the sequence  $\{r^n\}_{n=1}^{\infty}$  converges if  $-1 < r \leq 1$ , and divergent otherwise.

This is because  $\lim_{n \rightarrow \infty} r^n = \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \end{cases}$ ; the limit DNE for other values of  $r$ .

**Limit laws for sequences:** if  $\{a_n\}$  and  $\{b_n\}$  are convergent sequences, and  $C$  is a constant:

$$\cdot \lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$$

$$\cdot \lim_{n \rightarrow \infty} C \cdot a_n = C \cdot \lim_{n \rightarrow \infty} a_n ; \quad \lim_{n \rightarrow \infty} C = C.$$

$$\cdot \lim_{n \rightarrow \infty} a_n \cdot b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n ; \quad \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \left( \lim_{n \rightarrow \infty} a_n \right) / \left( \lim_{n \rightarrow \infty} b_n \right), \text{ assuming } \lim_{n \rightarrow \infty} b_n \neq 0.$$

$$\cdot \lim_{n \rightarrow \infty} (a_n)^p = \left( \lim_{n \rightarrow \infty} a_n \right)^p, \text{ if } p > 0 \text{ and } a_n > 0 \text{ for all } n.$$

## More examples of limits.

$$\textcircled{10} \lim_{n \rightarrow \infty} \frac{\ln(1-5n)}{\sin(\pi n)} \left( \frac{0}{0} \right) \xrightarrow{\text{L.H.}} \lim_{n \rightarrow \infty} \frac{\frac{1}{1-5n} \cdot \frac{5}{n^2}}{\cos(\pi n) \left( -\frac{\pi}{n^2} \right)} = \frac{-\frac{5}{\pi} \cdot 0}{-\pi \cdot \cos 0} = -\frac{5}{\pi}.$$

$$\textcircled{11} \lim_{n \rightarrow \infty} \frac{1}{4n} \ln\left(\frac{9}{n}\right) (0.\infty) = \lim_{n \rightarrow \infty} \frac{\ln 9 - \ln(n)}{4n} \left( \frac{\infty}{\infty} \right) \xrightarrow{\text{L.H.}} \lim_{n \rightarrow \infty} \frac{-\frac{1}{n}}{\frac{4}{n}} = 0$$

$$\textcircled{12} \lim_{n \rightarrow \infty} \sqrt[5n]{3n} = \lim_{n \rightarrow \infty} (3n)^{\frac{1}{5n}} (\infty^0). \text{ Let } y = (3n)^{\frac{1}{5n}}, \ln y = \frac{\ln(3n)}{5n}; \lim_{n \rightarrow \infty} \ln y = \lim_{n \rightarrow \infty} \frac{\ln(3n)}{5n} \left( \frac{\infty}{\infty} \right) \xrightarrow{\text{L.H.}} \lim_{n \rightarrow \infty} \frac{\frac{1}{3n}}{\frac{5}{5n}} = 0. \text{ Therefore } \lim_{n \rightarrow \infty} y = e^0 = 1.$$

$$\textcircled{13} \lim_{n \rightarrow \infty} \frac{8 + \frac{2}{n}}{(3n)^{1/n}}; \lim_{n \rightarrow \infty} 8 + \frac{2}{n} = 8; \lim_{n \rightarrow \infty} (3n)^{\frac{1}{n}} = 1. \text{ (use the method in Ex. 12).}$$

$$\text{Therefore } \lim_{n \rightarrow \infty} \frac{8 + \frac{2}{n}}{(3n)^{1/n}} = \frac{8}{1} = 8.$$

$$\textcircled{14} \lim_{n \rightarrow \infty} 4 \left(1 - \frac{1}{n^2}\right)^{7n} (1^\infty). \text{ Let } y = \left(1 - \frac{1}{n^2}\right)^{7n}, \ln(y) = 7n \ln\left(1 - \frac{1}{n^2}\right).$$

$$\text{Then } \lim_{n \rightarrow \infty} \ln(y) = \lim_{n \rightarrow \infty} \frac{\ln\left(1 - \frac{1}{n^2}\right)}{\frac{1}{7n}} \left( \frac{0}{0} \right) \xrightarrow{\text{L.H.}} \lim_{n \rightarrow \infty} \frac{\frac{1}{1-\frac{1}{n^2}} \cdot \frac{2}{n^3}}{-\frac{1}{7n^2}} = 0. \text{ Therefore}$$

$$\lim_{n \rightarrow \infty} y = e^0 = 1, \text{ and finally, } \lim_{n \rightarrow \infty} 4y = 4 \cdot 1 = 4.$$